Revisiting q-Derangement Numbers via Decorated Permutations

Kathy Q. Ji

Center for Applied Mathematics
Tianjin University
Tianjin 300072, P.R. China

kathyji@tju.edu.cn

Abstract. This note aims to provide a direct combinatorial proof of the Gessel–Reutenauer–Wachs formula for q-derangement numbers in the setting of decorated permutations, without using the q-binomial inversion formula. Decorated permutations, introduced by Postnikov in his study of the totally nonnegative Grassmannian, provide a natural framework for Chen's signed fixed-point model. Our proof is based on a major-index generating function for decorated permutations with a fixed number of signed fixed points, together with a sign-reversing and descent-set-preserving involution, thereby answering a question raised by Chen. This involution was discovered through human–AI collaboration.

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1 Introduction

We will follow the terminology and notation on permutations and partitions in Andrews [2] and Stanley [11]. Let $\mathfrak{S}_n$ denote the set of all permutations on $[n] := \{1, 2, \ldots, n\}$. An integer i with $1 \leq i \leq n$ is said to be a *fixed point* of $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$ if $\pi_i = i$, and a *nonfixed point* otherwise. *Derangements* are permutations with no fixed points. Let $\mathcal{D}_n$ be the set of all derangements in $\mathfrak{S}_n$ and let d_n denote the number of derangements in $\mathfrak{S}_n$. It is well-known that

$$d_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}. \tag{1.1}$$

In *An Undergraduate Course in Combinatorics*, William Y. C. Chen gave a simple combinatorial proof of (1.1) by considering permutations in which fixed points may be assigned a minus sign. It is worth noting that Chen's signed fixed-point model happens to coincide with decorated permutations, namely permutations in which each fixed point is assigned one of two colors.

Decorated permutations were introduced by Postnikov [10] in his study of the totally nonnegative Grassmannian. Corteel [6] and Williams [13] further studied decorated permutations in connection with permutation statistics such as weak excedances and alignments. For the definitions of weak excedances and alignments, see [6, 13]. More generally, Blitvić and Steingrímsson [3] introduced k-arrangements, namely permutations whose fixed points are colored with one of k colors; Thus decorated permutations are precisely 2-arrangements; see also Fu, Han, and Lin [9].

Let E_n be the set of decorated permutations on $[n]$. A signed fixed point is denoted by $\bar{i}$. For $0 \le k \le n$, let $E_{n,k}$ denote the subset of E_n with exactly k signed fixed points. Clearly,

$$|E_{n,k}| = \binom{n}{k}(n-k)! = \frac{n!}{k!}. \tag{1.2}$$

Consider the subset of E_n consisting of permutations with at least one fixed point, signed or unsigned. Let i be the minimum fixed point (in absolute value). Changing the sign of i gives a sign-reversing involution. After the cancellation, only ordinary derangements remain, which yields (1.1).

William Y. C. Chen posed the natural question of whether this signed fixed-point framework can also be applied to the q-derangement numbers; see [4] for details.

To state the formula for the q-derangement numbers, we first recall the definitions of the descent set and the major index on ordinary permutations. For $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, we say that $1 \le i \le n-1$ is a descent if $\pi_i > \pi_{i+1}$ and $1 \le i \le n-1$ is an ascent if $\pi_i < \pi_{i+1}$. The set of descents of π is called the descent set of π, denoted $\mathrm{Des}(\pi)$ and the number of its descents is called the descent number, denoted $\mathrm{des}(\pi)$. The major index of π, denoted $\mathrm{maj}(\pi)$, is defined to be the sum of its descents. To wit,

$$\mathrm{maj}(\pi) := \sum_{k \in \mathrm{Des}(\pi)} k.$$

The following formula due to MacMahon is well-known:

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{maj}(\pi)} = [n]_q! = [1]_q[2]_q \cdots [n]_q. \tag{1.3}$$

Here and in the sequel, $[0]_q! = 1$ and for a positive integer n, we define

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

The q-derangement numbers are defined by $d_0(q) = 1$ and for $n \ge 1$,

$$d_n(q) = \sum_{\sigma \in \mathcal{D}_n} q^{\mathrm{maj}(\sigma)}.$$

Gessel and Reutenauer [7] derived the following elegant formula for the q-derangement numbers, which they obtained as a consequence of the quasi-symmetric generating function encoding the descents and the cycle structure of permutations. Note that setting $q = 1$ in (1.4) recovers the classical derangement formula (1.1).

Theorem 1.1 (Gessel-Reutenauer-Wachs). *For $n \geq 1$,*

$$d_n(q) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \frac{[n]_q!}{[k]_q!}, \tag{1.4}$$

A combinatorial proof of (1.4) has been obtained by Wachs [12]. Let us first review the combinatorial settings of Wachs. Let $A = \{0 < a_1 < a_2 < \cdots < a_n\}$ and let $\mathfrak{S}_A$ denote the set of permutations of the set A. For $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_A$, the reduction of π is the permutation in $\mathfrak{S}_n$ by replacing each letter a_j by j. For example, $\pi = 9\,3\,8\,10\,12\,2\,7$ is the permutation of the set $A = \{2, 3, 7, 8, 9, 10, 12\}$, its reduction is $5\,2\,4\,6\,7\,1\,3$, which is a permutation in $\mathfrak{S}_7$.

Let $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$. The derangement part of π, denoted $dp(\pi)$, is the reduction of the subword of non-fixed points of π. Recall that π_i is called the non-fixed points of π if $\pi_i \neq i$. For example, let's take the following permutation in $\mathfrak{S}_9$:

$$\pi = 1\,5\,3\,7\,6\,2\,9\,8\,4, \tag{1.5}$$

there are three fixed points, which are $1, 3, 8$ and six non-fixed points: 5, 7, 6, 2, 9, 4. The reduction of non-fixed points of π is $3\,5\,4\,1\,6\,2$, so the derangement part of π is

$$dp(\pi) = 3\,5\,4\,1\,6\,2.$$

Wachs [12] established the following relation:

Proposition 1.2 (Wachs). *Let $0 \leq k \leq n$ and $\sigma \in \mathcal{D}_k$. We have*

$$\sum_{\substack{dp(\pi)=\sigma \\ \pi \in \mathfrak{S}_n}} q^{\mathrm{maj}(\pi)} = q^{\mathrm{maj}(\sigma)} \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{1.6}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

is the q-binomial coefficients.

Summing over all derangements $\pi \in \mathcal{D}_k$ and $0 \leq k \leq n$, and applying (1.3), we can deduce from (1.6) that

$$[n]! = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q d_k(q).$$

Thus (1.4) follows from the q-binomial inversion [1, Corollary 3.38].

In order to justify the relation (1.6), Wachs [12] found a bijection on $\mathfrak{S}_n$ by rearranging a permutation π according to *excedant* ($\pi_i > i$), fixed point, and *subcedant*

($\pi_i < i$). She showed that this bijection preserves the major index. Then the following result of Garsia-Gessel [8, Theorem 3.1] on shuffles of permutations is applied to establish Theorem 1.1.

Let $\pi \in \mathfrak{S}_j$ and $\delta \in \mathfrak{S}_k$ be two disjoint permutations, that is, permutations with no letters in common. We say that $\alpha \in \mathfrak{S}_{j+k}$ is a shuffle of π and δ if both π and δ are subsequences of α. The set of shuffles of π and δ is denoted $\pi \sqcup\!\sqcup\, \delta$. For example, let $\pi = 263$ and $\delta = 14$, we have

$$263 \sqcup\!\sqcup\, 14 = \{26314, 26134, 26143, 21463, 21634, 21643, 12463, 14263, 12634, 12643\}.$$

Theorem 1.3 (Garsia-Gessel). *Let $\pi \in \mathfrak{S}_m$ and $\delta \in \mathfrak{S}_n$ be two disjoint permutations. We have*

$$\sum_{\alpha \in \pi \sqcup\!\sqcup \delta} q^{\mathrm{maj}(\alpha)} = q^{\mathrm{maj}(\pi)+\mathrm{maj}(\delta)} \begin{bmatrix} n+m \\ n \end{bmatrix}_q. \tag{1.7}$$

Inspired by MacMahon's original proof of (1.3), Chen and Xu [5] present an alternative approach to Wachs' formula (1.6) based on the following reformulation in terms of labeled partitions:

$$\frac{1}{(q)_n} \sum_{\substack{\pi \in \mathfrak{S}_n \\ dp(\pi)=\sigma}} q^{\mathrm{maj}(\pi)} = \frac{1}{(q)_k (q)_{n-k}} q^{\mathrm{maj}(\sigma)}. \tag{1.8}$$

William Y. C. Chen raised the problem of finding a direct proof of Theorem 1.1 in the setting of decorated permutations, without using the q-binomial inversion formula. As a first step, Chen extended the descent set and the major index from ordinary permutations to decorated permutations in E_n. This is done by regarding a decorated permutation as a word in the alphabet

$$\bar{n} < \cdots < \bar{2} < \bar{1} < 1 < 2 < \cdots < n. \tag{1.9}$$

For example,

$$\mathrm{Des}(2\,1\,3\,\overline{4}\,6\,5\,\overline{7}) = \{1, 3, 5, 6\}, \quad \mathrm{maj}(2\,1\,3\,\overline{4}\,6\,5\,\overline{7}) = 15.$$

Chen [4] observed the following formula for the major-index generating function over decorated permutations with a fixed number of signed fixed points. When $q = 1$, this formula reduces to (1.2).

Theorem 1.4. *For $0 \le k \le n$, let $E_{n,k}$ denote the set of decorated permutations in E_n with exactly k signed fixed points. We have*

$$\sum_{\pi \in E_{n,k}} q^{\mathrm{maj}(\pi)} = q^{\binom{k}{2}} \frac{[n]_q!}{[k]_q!}. \tag{1.10}$$

As mentioned by Chen [4], Catherine Yan gave a proof of (1.10) using Foata's bijection between the inversion number and the major index, while Peter Guo gave a proof using Stanley's theory of P-partitions. In Section 2, we provide a simple direct proof based on MacMahons major index formula (1.3) and the Garsia–Gessel shuffle formula (1.7).

Chen further asked for a sign-reversing and descent-set-preserving involution on $G_n = E_n \setminus \mathcal{D}_n$, where $\mathcal{D}_n$ is regarded as the set of ordinary unsigned derangements embedded in E_n. More precisely,

Theorem 1.5. *Let G_n denote the set of decorated permutations on $[n]$ with at least one fixed point, either signed or unsigned. For $\pi \in G_n$, let $\mathrm{nfix}(\pi)$ denote the number of signed fixed points of π. We have*

$$\sum_{\pi \in G_n} (-1)^{\mathrm{nfix}(\pi)} q^{\mathrm{maj}(\pi)} = 0. \tag{1.11}$$

Combining Theorems 1.4 and 1.5 immediately yields Theorem 1.1. In Section 3, we construct the desired sign-reversing and descent-set-preserving involution by extending Wachs' setting to decorated permutations.

2 Proof of Theorem 1.4

Given $0 \le k \le n$, let $\mathfrak{S}(k+1, \dots, n)$ denote the set of permutations on $\{k+1, \dots, n\}$, and let $\delta_k = (k, k-1, \dots 1)$ with the convention that δ_0 is the empty word. There is a canonical descent-preserving bijection

$$\phi\colon E_{n,k} \longrightarrow \cup_{\beta \in \mathfrak{S}(k+1,\dots,n)} \delta_k \sqcup \beta$$

such that $\mathrm{Des}(\phi(\pi)) = \mathrm{Des}(\pi)$ for every $\pi \in E_{n,k}$.

Using this natural bijection ϕ, we give a short direct proof of Theorem 1.4, relying on MacMahons major index formula (1.3) and the Garsia-Gessel shuffle identity (1.7).

Proof of Theorem 1.4. By the descent-set-preserving bijection ϕ, we derive that

$$\begin{aligned}
\sum_{\pi \in E_{n,k}} q^{\mathrm{maj}(\pi)} &= \sum_{\beta \in \mathfrak{S}\{k+1,\dots,n\}} \sum_{\phi(\pi) \in \delta_k \sqcup \beta} q^{\mathrm{maj}(\phi(\pi))} \\
&\overset{(1.7)}{=} \sum_{\beta \in \mathfrak{S}\{k+1,\dots,n\}} q^{\mathrm{maj}(\delta_k) + \mathrm{maj}(\beta)} \begin{bmatrix} n \\ k \end{bmatrix}_q \\
&= q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\beta \in \mathfrak{S}\{k+1,\dots,n\}} q^{\mathrm{maj}(\beta)} \\
&\overset{(1.3)}{=} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [n-k]_q! = q^{\binom{k}{2}} \frac{[n]_q!}{[k]_q!},
\end{aligned}$$

as desired. ∎

3 An involution

In this section, we aim to prove Theorem 1.5 by constructing the sign-reversing and descent-set-preserving involution on decorated permutations in Wachs setting. In fact, this involution establishes the following stronger refined identity:

Theorem 3.1. *For $0 \le k \le n-1$, let $\sigma \in \mathcal{D}_k$, we have*

$$\sum_{\pi \in E_n(\sigma)} (-1)^{\mathrm{nfix}(\pi)} z^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} = 0. \tag{3.1}$$

Summing the above identity over all $\sigma \in \mathcal{D}_k$ for $0 \le k \le n-1$ immediately recovers Theorem 1.5.

We first extend Wachs combinatorial setting to decorated permutations. For $\pi = \pi_1 \dots \pi_n \in E_n$, we say that a letter π_i of π is an excedant (resp. subcedant) of π if $\pi_i > i$ (resp. $\pi_i < i$). Let $s(\pi)$ and $e(\pi)$ denote the numbers of subcedants and excedants of π respectively. We now fix n and let $k \le n$. *The map* ψ_n is defined as follows: Let $\pi \in E_k$, then $\psi_n(\pi) = \tilde{\pi}$ is obtained from π by replacing its ith smallest subcedant π_j by i, $i = 1, 2, \dots, s(\pi)$, its ith smallest (in absolute value) fixed point π_j by $(\mathrm{sgn}(\pi_j))(s(\pi)+i)$, $i = 1, 2, \dots, k - s(\pi) - e(\pi)$, and its ith largest excedant by $n - i + 1$, $i = 1, 2, \dots, e(\pi)$.

For example, let $\pi = 5\,8\,\bar{3}\,1\,6\,4\,7\,2 \in E_8$. We see that $s(\pi) = 3$, $e(\pi) = 3$ and $\sigma = dp(\pi) = 4\,6\,1\,5\,3\,2$. Then

$$\psi_8(\sigma) = 6\,8\,1\,7\,3\,2, \quad \psi_8(\pi) = 6\,8\,\bar{4}\,1\,7\,3\,5\,2.$$

For $\sigma \in \mathcal{D}_k$, let $E_n(\sigma)$ denote the set of permutations π in E_n such that $dp(\pi) = \sigma$, and let $\Gamma(\sigma)$ be the set of all signed sequences $(a_1, \dots, a_{n-k})$, where $a_i = s(\sigma) + i$ or $\overline{s(\sigma) + i}$. We have the following consequence:

Lemma 3.2. *For $0 \le k \le n$, let $\sigma \in \mathcal{D}_k$ and set $\tilde{\sigma} = \psi_n(\sigma)$. Let $\mathbf{Sh}(\tilde{\sigma}, \Gamma(\sigma))$ denote the shuffles of $\tilde{\sigma}$ and $\gamma \in \Gamma(\sigma)$. Then the map ψ_n provides a bijection between $E_n(\sigma)$ and $\mathbf{Sh}(\tilde{\sigma}, \Gamma(\sigma))$. Moreover, for $\pi \in E_n(\sigma)$, we have $\psi_n(\pi) \in \mathbf{Sh}(\widetilde{\sigma}, \gamma)$ for some $\gamma \in \Gamma(\sigma)$ such that*

$$\mathrm{Des}(\pi) = \mathrm{Des}(\psi_n(\pi)), \quad \mathrm{nfix}(\pi) = \mathrm{nfix}(\gamma). \tag{3.2}$$

Proof. By the construction of ψ_n, it is straightforward to verify that ψ_n gives a bijection between $E_n(\sigma)$ and $\mathbf{Sh}(\widetilde{\sigma}, \Gamma(\sigma))$. It remains to prove the two relations in (3.2). Wachs proved that the corresponding map preserves descent sets in the unsigned case. We claim that the signed extension considered here also preserves descent sets. Indeed, when two adjacent letters are both positive, the assertion is exactly Wachs' result. If one of the adjacent letters is a minus-signed fixed point, then this letter is smaller than every positive letter with respect to the order (1.9). Thus the descent status of such an adjacent pair is unchanged under ψ_n. Finally, if both adjacent letters are minus-signed fixed points, then their relative order is reversed by the above signed order, and the same reversal is reflected in the order of the fixed-point word $\gamma \in \Gamma(\sigma)$. Hence ψ_n preserves descent sets, that is, $\mathrm{Des}(\pi) = \mathrm{Des}(\psi_n(\pi))$. Moreover, ψ_n clearly preserves the number of minus-signed fixed points. Therefore, $\mathrm{nfix}(\pi) = \mathrm{nfix}(\gamma)$. This completes the proof. ∎

Armed with Lemma 3.2, we are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Given $\sigma \in \mathcal{D}_k$ for $0 \le k \le n-1$, it suffices to construct an involution Ψ on $E_n(\sigma)$ such that, for every $\pi \in E_n(\sigma)$,

$$(-1)^{\mathrm{nfix}(\Psi(\pi))} = -(-1)^{\mathrm{nfix}(\pi)}, \qquad \mathrm{Des}(\Psi(\pi)) = \mathrm{Des}(\pi). \tag{3.3}$$

Let $\pi \in E_n(\sigma)$ and set $\tilde{\pi} = \psi_n(\pi)$. By Lemma 3.2, we see that $\tilde{\pi} \in \mathbf{Sh}(\tilde{\sigma}, \Gamma(\sigma))$, where $\tilde{\sigma} = \psi_n(\sigma)$. Thus we may write $\tilde{\pi} = \tilde{\sigma} \sqcup \gamma$, where $\gamma = \gamma_1 \dots \gamma_{n-k} \in \Gamma(\sigma)$. Note that $\gamma_1 = \overline{s(\sigma)+1}$ or $s(\sigma)+1$. Assume that $\tilde{\pi} = \tilde{\pi}_1 \dots \tilde{\pi}_n$, where $\tilde{\pi}_j = \gamma_1$.

We now define a word $\widetilde{\tau}$ by changing the sign of the distinguished letter γ_1 and, when necessary, sliding it across a maximal string of letters in the interval $(\overline{s(\sigma)+1}, s(\sigma)+1)$ so that the descent set is preserved. Note that all intervals below are taken with respect to the order given as (1.9). Finally, define

$$\Psi(\pi) = \tau = \psi_n^{-1}(\widetilde{\tau}).$$

We distinguish two cases:

1. Suppose that $\gamma_1 = \overline{s(\sigma)+1}$, that is $\tilde{\pi}_j = \overline{s(\sigma)+1}$. We consider the following two subcases:

   (a) If neither $\tilde{\pi}_{j-1}$ or $\tilde{\pi}_{j+1}$ belongs to $(\overline{s(\sigma)+1}, s(\sigma)+1)$, then we define $\widetilde{\tau}$ by replacing $\overline{s(\sigma)+1}$ in $\tilde{\pi}$ by $s(\sigma)+1$:

   $$\widetilde{\tau} = \widetilde{\pi}_1 \cdots \widetilde{\pi}_{j-1}\, (s(\sigma)+1)\, \widetilde{\pi}_{j+1} \cdots \widetilde{\pi}_n.$$

   .

   (b) Suppose that $\tilde{\pi}_{j-1} \in (\overline{s(\sigma)+1}, s(\sigma)+1)$. There are two cases:

      i. If $\tilde{\pi}_{j-1} < \tilde{\pi}_{j+1}$, then choose maximal r such that $1 \le r \le j-1$ and $\widetilde{\pi}_{j-i} \in (\widetilde{\pi}_{j-i+1}, s(\sigma)+1)$ for $1 \le i \le r$. Equivalently, $\widetilde{\pi}_{j-r-1} \notin (\widetilde{\pi}_{j-r}, s(\sigma)+1)$. Define

      $$\widetilde{\tau} = \widetilde{\pi}_1 \cdots \widetilde{\pi}_{j-r-1}\, (s(\sigma)+1)\, \widetilde{\pi}_{j-r} \cdots \widetilde{\pi}_{j-1} \widetilde{\pi}_{j+1} \cdots \widetilde{\pi}_n.$$

ii. If $\tilde{\pi}_{j-1} > \tilde{\pi}_{j+1}$, then choose maximal r such that $1 \le r \le n-j$ and $\widetilde{\pi}_{j+i} \in (\widetilde{\pi}_{j+i-1}, s(\sigma)+1)$ for $1 \le i \le r$. Equivalently, $\widetilde{\pi}_{j+r+1} \notin (\widetilde{\pi}_{j+r}, s(\sigma)+1)$. Define

$$\widetilde{\tau} = \widetilde{\pi}_1 \cdots \widetilde{\pi}_{j-1}\widetilde{\pi}_{j+1} \cdots \widetilde{\pi}_{j+r}\,(s(\sigma)+1)\,\widetilde{\pi}_{j+r+1} \cdots \widetilde{\pi}_n.$$

(c) Suppose that $\tilde{\pi}_{j+1} \in (\overline{s(\sigma)+1}, s(\sigma)+1)$.

i. If $\tilde{\pi}_{j-1} < \tilde{\pi}_{j+1}$, and so $\tilde{\pi}_{j-1} \in (\overline{s(\sigma)+1}, s(\sigma)+1)$, then we apply the construction of Case 1 (b) (i) in the leftward direction.

ii. If $\tilde{\pi}_{j-1} > \tilde{\pi}_{j+1}$, then we apply the construction of Case 1 (b) (ii) in the rightward direction.

2. Suppose that $\gamma_1 = s(\sigma)+1$, that is $\tilde{\pi}_j = s(\sigma)+1$.

(a) If neither $\tilde{\pi}_{j-1}$ nor $\tilde{\pi}_{j+1}$ belongs to $(\overline{s(\sigma)+1}, s(\sigma)+1)$, then we define $\widetilde{\tau}$ by replacing $s(\sigma)+1$ in $\widetilde{\pi}$ by $\overline{s(\sigma)+1}$:

$$\widetilde{\tau} = \widetilde{\pi}_1 \cdots \widetilde{\pi}_{j-1}\,\overline{s(\sigma)+1}\,\widetilde{\pi}_{j+1} \cdots \widetilde{\pi}_n.$$

(b) Suppose that $\tilde{\pi}_{j-1} \in (\overline{s(\sigma)+1}, s(\sigma)+1)$.

i. If $\tilde{\pi}_{j-1} < \tilde{\pi}_{j+1}$ and $\tilde{\pi}_{j+1} \in (\overline{s(\sigma)+1}, s(\sigma)+1)$, then choose maximal r such that $1 \le r \le n-j$ and $\tilde{\pi}_{j+i} \in (\overline{s(\sigma)+1}, \tilde{\pi}_{j+i-1})$ for $1 \le i \le r$. Equivalently, $\tilde{\pi}_{j+r+1} \notin (\overline{s(\sigma)+1}, \tilde{\pi}_{j+r})$. Define

$$\tilde{\tau} = \tilde{\pi}_1 \;\cdots \tilde{\pi}_{j-1}\,\tilde{\pi}_{j+1} \;\cdots\; \tilde{\pi}_{j+r}\,\overline{s(\sigma)+1}\,\tilde{\pi}_{j+r+1} \;\cdots \tilde{\pi}_n.$$

ii. If $\tilde{\pi}_{j-1} < \tilde{\pi}_{j+1}$ and $\tilde{\pi}_{j+1} \notin (\overline{s(\sigma)+1}, s(\sigma)+1)$, then choose maximal r such that $1 \le r \le j$ and $\tilde{\pi}_{j-i} \in (\overline{s(\sigma)+1}, \tilde{\pi}_{j-i+1})$ for $1 \le i \le r$. Equivalently, $\tilde{\pi}_{j-r-1} \notin (\overline{s(\sigma)+1}, \tilde{\pi}_{j-r})$. Define

$$\tilde{\tau} = \tilde{\pi}_1 \;\ldots \tilde{\pi}_{j-r-1}\,(\overline{s(\sigma)+1})\,\tilde{\pi}_{j-r} \;\ldots\; \tilde{\pi}_{j-1}\,\tilde{\pi}_{j+1} \;\ldots\; \tilde{\pi}_n.$$

iii. If $\tilde{\pi}_{j-1} > \tilde{\pi}_{j+1}$, then we apply the construction of Case 2 (b) (ii) in the leftward direction.

(c) Suppose that $\tilde{\pi}_{j+1} \in (\overline{s(\sigma)+1}, s(\sigma)+1)$.

i. If $\tilde{\pi}_{j-1} < \tilde{\pi}_{j+1}$, then we apply the construction of Case 2 (b) (i) in the rightward direction.

ii. If $\tilde{\pi}_{j-1} > \tilde{\pi}_{j+1}$ and $\tilde{\pi}_{j-1} \in (\overline{s(\sigma)+1}, s(\sigma)+1)$, then we apply the construction of Case 2 (b) (ii) in the leftward direction.

iii. If $\tilde{\pi}_{j-1} > \tilde{\pi}_{j+1}$ and $\tilde{\pi}_{j-1} \notin (\overline{s(\sigma)+1}, s(\sigma)+1)$, then we apply the construction of Case 2 (b) (i) in the rightward direction.

In each of the above cases, the word $\widetilde{\tau}$ is obtained from $\widetilde{\pi}$ by changing exactly one letter, namely $\overline{s(\sigma)+1}$ to $s(\sigma)+1$ or $s(\sigma)+1$ to $\overline{s(\sigma)+1}$, and possibly moving this letter across a maximal string of letters lying in the interval $(\overline{s(\sigma)+1}, s(\sigma)+1)$. A direct comparison of adjacent letters shows that $\mathrm{Des}(\widetilde{\tau}) = \mathrm{Des}(\widetilde{\pi})$. Moreover, $\widetilde{\tau} \in \mathbf{Sh}(\widetilde{\sigma}, \Gamma(\sigma))$. Hence, by Lemma 3.2, $\mathrm{Des}(\tau) = \mathrm{Des}(\pi)$. Since exactly one sign has been changed, we also have $\mathrm{nfix}(\tau) \equiv \mathrm{nfix}(\pi) + 1 \pmod 2$. Thus Ψ is sign-reversing and descent-set-preserving.

Finally, applying the same rule to τ changes the distinguished letter back and slides it to its original position. Therefore, $\Psi(\Psi(\pi)) = \pi$. This proves that Ψ is the desired involution and completes the proof. ∎

For example, let $\pi = 1\,6\,\bar{3}\,5\,8\,2\,7\,4 \in E_8$. Then $\sigma = dp(\pi) = 4\,6\,1\,5\,3\,2$. We have

$$\widetilde{\sigma} = \psi_8(\sigma) = 6\,8\,1\,7\,3\,2$$

and

$$\widetilde{\pi} = \psi_8(\pi) = 6\,8\,\bar{4}\,1\,7\,3\,5\,2 \in \mathbf{Sh}(\widetilde{\sigma}, \gamma), \text{ where } \gamma = \bar{4}\,5.$$

It can be checked that $\widetilde{\pi}$ falls into Case 1(c) (ii), and the construction gives

$$\widetilde{\tau} = 6\,8\,1\,4\,7\,3\,5\,2.$$

Thus

$$\tau = \Psi(\pi) = 5\,8\,1\,4\,6\,3\,7\,2.$$

Hence $\tau \in E_8(\sigma)$, since $dp(\tau) = dp(\pi) = 4\,6\,1\,5\,3\,2$. Moreover, $(-1)^{\mathrm{nfix}(\tau)} = -(-1)^{\mathrm{nfix}(\pi)}$, $\mathrm{Des}(\pi) = \mathrm{Des}(\tau) = \{2, 5, 7\}$, and $\Psi(\tau) = \pi$.

The following table illustrates the involution on $G_3 = E_3 \setminus \mathcal{D}_3$:

Des	Pairings	
$\varnothing$	$123 \leftrightarrow \bar{1}23$	
$\{1\}$	$213 \leftrightarrow 3\bar{2}1$,	$1\bar{2}3 \leftrightarrow \bar{1}\bar{2}3$
$\{2\}$	$132 \leftrightarrow \bar{1}32$,	$12\bar{3} \leftrightarrow \bar{1}2\bar{3}$
$\{1,2\}$	$321 \leftrightarrow 21\bar{3}$,	$1\bar{2}\bar{3} \leftrightarrow \bar{1}\bar{2}\bar{3}$

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